Metastabilities in the degenerated phase of the two-component model

Katarzyna Sznajd-Weron*
Institute of Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland
(Received 22 March 2005; published 8 August 2005)

In previous papers, we have introduced a new dynamical model of Ising spins, namely the two-component (TC) model. Using the Boltzmann factor, in this paper we introduce parameter T to the model. This is the standard method for introducing temperature. However, since we have not defined the energy for the TC model, but only the disagreement function, we will not call this parameter temperature. We will investigate the system in its degenerated phase, which consists of four qualitatively different steady states at $T=0$. We will show that for $T>0$, three of these steady states become metastable, and that above $T=T^*$ they become unstable. In the range $0<T<T^*$, the evolution of the system consists of relatively long stagnation periods, where the system remains in one of the metastable states, and rapid transition periods, where the system goes from one metastable state to another. In this range, the distribution function of waiting times needed to reach one of the states (steady for $T=0$) has an exponential tail with a T-dependent exponent.

DOI: 10.1103/PhysRevE.72.026109 PACS number(s): 05.50.+q, 05.45.−a, 05.10.−a

I. INTRODUCTION

Controlling the transitions from metastable states to equilibrium in the stochastic dynamics of lattice spin systems at low temperatures has been and still is a subject of considerable interest in statistical mechanics [1–10]. The concepts of metastability date back to the early days of thermodynamics and recently have been recalled using the language of the Landau theory for a ferromagnet by Godrèche and Luck [8]. Within the mean-field approach at fixed temperature $T<T_c$, there are three types of steady states: stable (global minimum of the free energy), metastable (local minimum of the free energy), and unstable (maximum of the free energy). It was realized in the 1960s that metastability is an artefact of the mean-field approximation [8]. From the 1970s on, a variety of complex systems have been shown to possess many metastable states at low enough temperature, and metastable states have thus been rediscovered under various names and with various definitions. The first rigorous approach traces back to [11], where metastability is characterized as a slow evolution of the averages over the process towards the stable equilibrium value. Recently, it has been found that even the simple Ising ferromagnet has a large number of metastable states with respect to Glauber spin-flip dynamics [9,10]. Therefore, at zero temperature the system could get stuck forever in one of these states. There appears to be a nonzero probability that the square lattice system freezes into a stripe configuration. At $T=0$, metastable states in this dynamics have an infinite lifetime that can prevent the equilibrium ground state from being reached. This is the basic reason why the dynamics at $T=0$ is different from that of small positive temperature [9,10]. For $T\to 0$, a system that enters a metastable state can escape and the true equilibrium state is eventually reached. The zero-temperature dynamics of simple models such as Ising ferromagnets provides an alternative to the mean-field situation, interesting examples of dynamical systems which lead to zero-temperature metastability. Recently, extensive results were obtained in one [6] and two dimensions (the square and honeycomb lattices) [2].

The single-spin-flip zero-temperature Glauber dynamics is a descent dynamics (i.e., every move strictly decreases the total energy). Recently, a new nondescent single-spin-flip dynamics has been introduced in the so called two-component (TC) model [12,13]. In this model, instead of the energy, the disagreement function has been introduced. In the TC model, every move decreases the local disagreement function, but can increase the global one [13]. Monte Carlo simulations and analytical reasoning have shown that on the square lattice (two-dimensional system), the TC model depending on two coefficients (J_1,J_2) can eventually lead the system to one of four phases: A (in which four qualitatively different steady states exist), B (ferromagnetic), C (with two different steady states), and D (antiferromagnetic). In one dimension, degeneration of steady states is lower—it disappears in phase C and there is double degeneration in phase A. Interestingly, in the TC model a degeneration of the steady state has been possible even if both coupling constants have been greater or smaller than zero, i.e., $J_2/J_1>0$, while in the ANNNI (axial next-nearest-neighbor Ising) [14] model, coupling constants need to have opposite signs in order to obtain degeneration, i.e., $J_2/J_1<0$.

In this paper, we concentrate on the most interesting, degenerated phase A of the TC model. We introduce parameter T to the model and show that for $T>0$, three of four steady states are metastable and only one (ferromagnetic) is stable. We show that metastable states in this dynamics have an infinite lifetime for $T=0$, while for small $T>0$ a system that enters a metastable state can escape from this state and the stable ferromagnetic state is eventually reached. Using Monte Carlo simulations we find the value of $T=T^*$, below which the system enters metastable states and above which it immediately goes to the stable steady state. We show that for $0<T<T^*$, the system remains for a relatively long time in one of the metastable states and then suddenly switches to

*Electronic address: kweron@ift.uni.wroc.pl; URL: http://www.ift.uni.wroc.pl/~kweron

another metastable state, eventually reaching the final stable steady state. In contrast to the Ising system under Glauber dynamics, the system described by the TC model has only a finite number of metastable states with four qualitatively different structures. In the next section, we briefly recall the idea of the TC model, show all possible types of steady states in degenerated phase A, and introduce parameter T to the model.

II. THE TC MODEL

The TC model [12,13] has been proposed to generalize the Sznajd model (for a review, see [15]), which was aimed at describing global social phenomena (sociology) by local social interactions (described by social psychology). The crucial difference of the Sznajd model compared to other Ising-type models is that information flows outward.

Because of the importance of this feature, we have decided to introduce a generalized model which kept our old dynamics (the outflow of information) but introduced a function controlling whether a spin should be flipped or not. The resulting model consists of the following two components (hence the name TC model) [12].

Dynamics: The information flows outward, i.e., a pair of spins S_i and S_{i+1} is chosen to change their nearest neighbors.

Disagreement function: The change of spins is controlled by a certain function (based on the ANNNI Hamiltonian [14]), which is locally minimized.

In one dimension [12], simulations and analytical reasoning show that the TC model depending on two interaction coefficients can eventually lead the system to one of four phases: (A) degenerated, in which two qualitatively different steady states exist—a ferromagnetic steady state is equally probable with an antiferromagnetic steady state, (B) ferromagnetic, (C) antiphase (2,2), and (D) antiferromagnetic.

In two dimensions [13], the TC model can also eventually lead the system to one of four phases: (A) degenerated, in which four (instead of two) qualitatively different steady states exist, (B) ferromagnetic, (C) doubly degenerated (instead of no degeneration), and (D) antiferromagnetic.

In this paper, we deal with the two-dimensional version of the TC model [13]. We investigate a system of Ising spins on a square lattice $L\times L$. The one-dimensional rule is applied to each of the four chains of four spins each, centered about two horizontal and two vertical pairs of light balls in Fig. 1. The algorithm is the following.

(i) Choose at random a spin, e.g., $S_{i,j}$, which defines a 2×2 box of spins $(S_{i,j},S_{i,j+1},S_{i+1,j},S_{i+1,j+1})$—light balls in Fig. 1.

(ii) Calculate the disagreement function for each of the eight nearest neighbors of the box defined in point 1 (chessboard colored balls in Fig. 1), e.g., for $S_{i-1,j}$,

$$E_{i-1,j} = -J_1 S_{i-1,j}S_{i,j} - J_2 S_{i-1,j}S_{i+1,j}. \quad (1)$$

(iii) Calculate the disagreement function for each of the eight nearest neighbors of the box in the case of a flipped spin, e.g., for $S_{i-1,j}$,

$$E'_{i-1,j} = J_1 S_{i-1,j}S_{i,j} + J_2 S_{i-1,j}S_{i+1,j}. \quad (2)$$

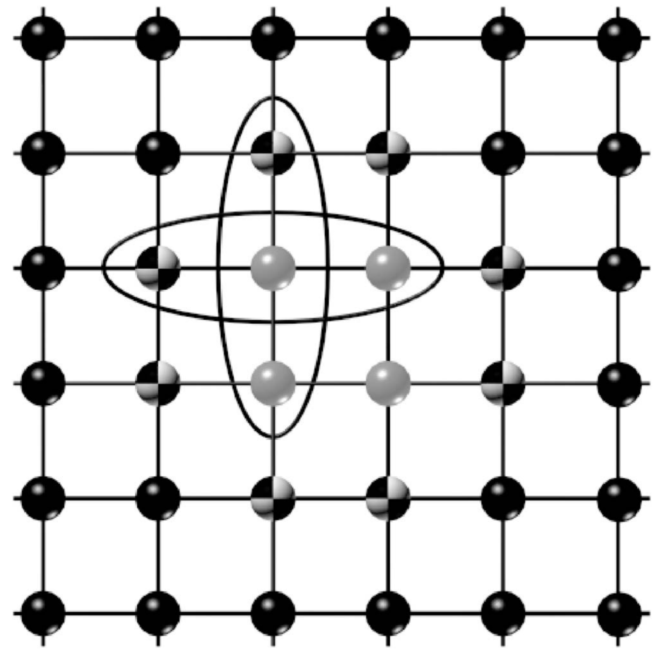

FIG. 1. Transformation of the one-dimensional TC model to two dimensions. The one-dimensional rule is applied to each of the four chains of the 2×2 box (light balls).

(iv) For each of the eight spins, check the difference in the disagreement function, e.g., for $S_{i-1,j}$: $\Delta E_{i-1,j}=E'_{i-1,j}-E_{i-1,j}$. If it is smaller than zero then flip the spin (e.g., $S_{i-1,j}$), otherwise leave it unchanged.

Summarizing, in the original two-dimensional TC model the spin $S_{i,j}$ is flipped with probability

$$p_{i,j} = 1 \text{ if } \Delta E_{i,j} < 0,$$

$$p_{i,j} = 0 \quad \text{otherwise}. \quad (3)$$

For such a model, we have shown [13] that in the case of $|J_1|<J_2$, four qualitatively different steady states are possible (see Fig. 2). It has been shown that even if we always start from exactly the same random initial state, we can reach each of the possible steady states presented in Fig. 2. One could guess that the probability of reaching a certain steady state depends on the disagreement function of the state, because the disagreement function plays the role of energy in the TC model. In a previous paper [13] we have defined the global disagreement function as

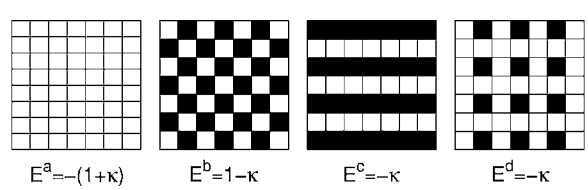

FIG. 2. All possible steady states with their values of the global disagreement function E (where $\kappa=J_2/J_1$) for the two-dimensional TC model in the case of $|J_1|<J_2$.

TABLE I. The global disagreement function and probabilities of reaching a given steady state for $T=0$ obtained by a simple counting of the possible equivalent configurations. Results agree with those obtained from Monte Carlo simulations [13].

Type	E	$E(\kappa=2)$	Probability
(a)	$-(1+\kappa)$	-3	1/8
(b)	$1-\kappa$	-1	1/8
(c)	$-\kappa$	-2	1/4
(d)	$-\kappa$	-2	1/2

$$E=\frac{1}{N}\sum_{i,j}E_{i,j}, \tag{4}$$

where N is the number of spins in the system and $E_{i,j}$, is the local disagreement function,

$$E_{i,j}=-J_1S_{i,j}S_{i+1,j}-J_2S_{i,j}S_{i+2,j}=-J_1S_{i,j}(S_{i+1,j}+\kappa S_{i+2,j}), \tag{5}$$

where $\kappa=J_2/J_1>1$. From now on we set $J_1=1$, using a simple expression for the disagreement function,

$$E_{i,j}=-S_{i,j}(S_{i+1,j}+\kappa S_{i+2,j}), \tag{6}$$

and we investigate our model in degenerated phase A, i.e., for $|\kappa|>1$.

The global disagreement function can be calculated easily for each possible steady state (see Fig. 2 and Table I). It has been shown that in the TC model, the probability of the concrete steady state does not depend on its global disagreement function, but only on the number of equivalent configurations connected with each type of the steady state (see Table I). For example, there are only two configurations for the ferromagnet—all spins up or all spins down, but four different configurations for type (c) and eight configurations for type (d).

In this paper, we introduce a new parameter T to the TC model in the following way: we flip spin $S_{i,j}$ with probability

$$p_{i,j}=\exp\left(\frac{-\Delta E_{i,j}}{T}\right). \tag{7}$$

This is the standard way of introducing temperature to the system. However, since there is no energy defined for the TC model, but only the disagreement function, we do not call T temperature.

In the next section, we present Monte Carlo results for $\kappa=2$. We have performed simulations on the square lattice $L\times L$ with periodic boundary conditions for several lattice sizes (from $L=20$ to $L=100$). The averaging was done usually over 10^4 samples.

III. SIMULATION RESULTS

Let us first look at the time evolution of the magnetization, defined as

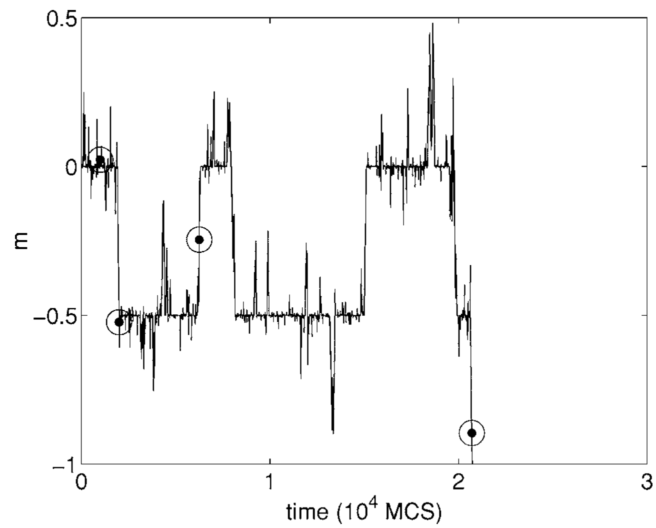

FIG. 3. A sample evolution of the magnetization for the system on the square lattice 60×60 with $T=1.9$. Configurations of the system at points denoted by circles are presented in Fig. 4. (MCS stands for Monte Carlo steps.)

$$m(t)=\frac{1}{L^2}\sum_{i,j}S_{i,j}(t). \tag{8}$$

We start from the antiferromagnetic state denoted by (b) in Fig. 2. We expect that for $T>0$, the antiferromagnetic state has the lowest stability, because its global disagreement function has the largest value among states (a)–(d). In Fig. 3, the time evolution of the magnetization for $T=0.19$ and $L=60$ is presented. It is seen that initially the system fluctuates around state (b) but after some time it rapidly goes out from this state to state (d). Again the system fluctuates for some time in state (d) and then rapidly goes to state (c) and so on. Once the system reaches the antiferromagnetic state, it stays in this state forever. This is understandable since for the ferromagnetic state [denoted by (a)] the global disagreement function has the lowest value among states (a)–(d). States (b)–(d) are now recognized as metastable states and only the ferromagnetic state is stable. Let us recall that for $T=0$ all four states have been recognized as stable steady states [13]. In Fig. 4, we have presented snapshots for these moments that are denoted by circles in Fig. 3. One can see the coexistence of several types of states.

The evolution presented in Fig. 3 is of course a sample one. In this example, the first reached state (steady for $T=0$, let us call it a trap), after escape from the initial antiferromagnetic state, was the one of type (d). However, it happens that the first trap is different. What is the probability of reaching a given type of state as the first one?

For $T=0$, the probabilities of reaching a certain steady state from a random initial state are given in Table I. The total number of all possible steady states is 16 [2 of type (a) +2 of type (b) +4 of type (c) +8 of type (d)]. For $T=0$, each of these 16 states is equally probable. On the other hand, we expect that above T^* the probability of state (a) is 1 and of all others is zero. Thus, the probability of reaching a given steady state, as the first in which the system is trapped for

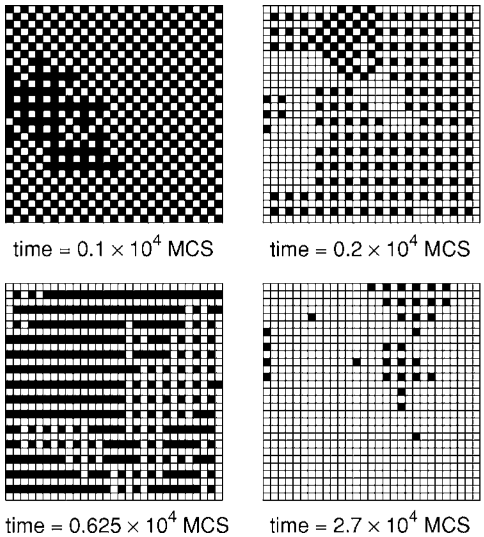

FIG. 4. The dynamical evolution of the system on the square lattice 60×60 with $T=0.19$ and the antifferomagnetic initial state. The snapshots are taken at times denoted by circles in Fig. 3.

some time, should be T-dependent. Moreover, the initial configuration might also influence this probability. In Figs. 5 and 6, the probabilities of the first trap as a function of T, for the square lattice 80×80, are presented. In Fig. 5, the initial state is random, while in Fig. 6 the initial state is antiferromagnetic. If we look at Fig. 5, we see that for $T\rightarrow0$, probability values indeed agree with those in Table I. However, above T^{*}, the ferromagnetic stable steady state is almost always reached as the first one. Simulations for antiferromagnetic initial conditions show that for low values of T, the system remains antiferromagnetic. With increasing T, remaining in the initial state becomes less probable and the probability of reaching another state increases. For $T>T^{*}$, the ferromagnetic state is almost always reached, analogously with random initial conditions.

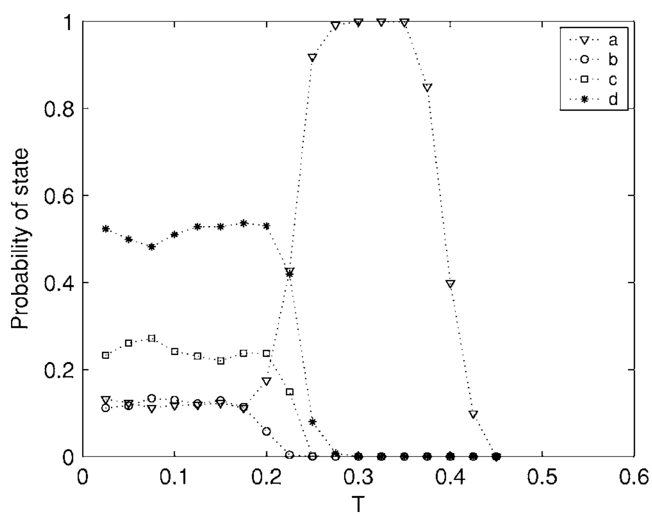

FIG. 5. Probabilities of the first trap as a function of T for the square lattice 80×80. Initial state is random. It is seen that above a certain value of $T=T^{*}$, the system goes directly to the stable ferromagnetic state and states (b)–(d) become unstable.

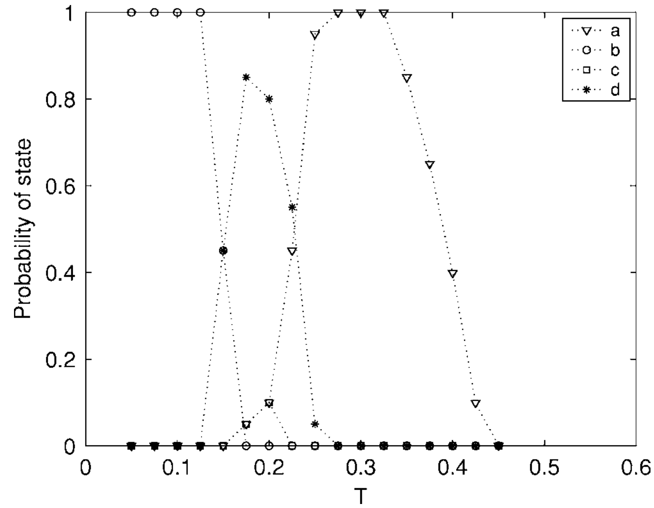

FIG. 6. Probabilities of the first trap as a function of T for the square lattice 80×80. Initial state is antiferromagnetic. It is seen that above a certain value of $T=T^{*}$, the system goes directly to the stable ferromagnetic state and states (b)–(d) become unstable.

To estimate the value of T^{*}, we have performed Monte Carlo simulations for several lattice sizes and we have found (see Fig. 7) that

$$T^{*}\approx0.19 \quad \text{for} \quad L\rightarrow\infty. \tag{9}$$

Now we look at the distribution function of waiting times to reach the first trap. For $0<T<T^{*}$, this distribution has an exponential tail with a T-dependent exponent (see Fig. 8). For $T=0$, the distribution of waiting times also has an exponential tail [16]. Mean saturation time is presented in the inset. One can see that it grows for large T. This is an anticipated behavior, because one can expect that for large T, the only stable steady state is the disordered one. This expectation is confirmed by the simulations (see Figs. 5 and 6) and by the mean-field approach, which will be presented in the next section.

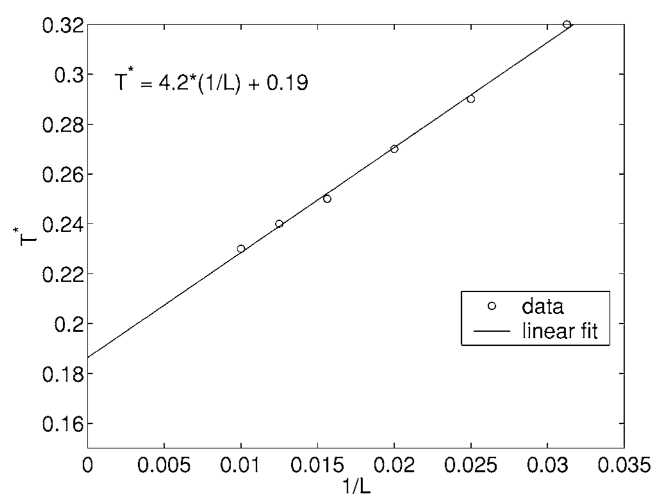

FIG. 7. Dependence between T^{*} and lattice size L.

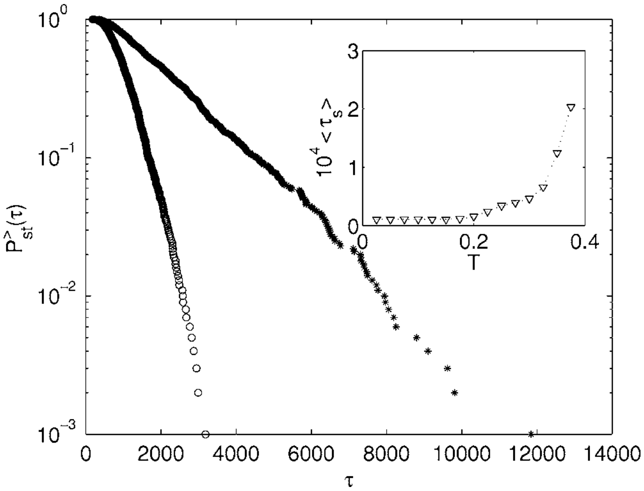

FIG. 8. Probabilities of reaching the trap (one of four presented in Fig. 2) in time larger than τ on the square lattice 80×80 for $T=1.5$ (○) and $T=2.25$ (*). Mean saturation time $\langle\tau_{\rm st}\rangle$ to reach the trap is presented in the inset.

IV. MEAN-FIELD APPROACH

The magnetization m in the system of N spins can change in time due to the following two events:

(i) If $S=-1\rightarrow S=1$, then the magnetization increases by $2/N$.

(ii) If $S=1\rightarrow S=-1$, then the magnetization decreases by $2/N$.

If we denote by N_+ the number of up spins and by N_- the number of down spins, then we can define the magnetization as

$$m=\frac{N_+-N_-}{N}. \tag{10}$$

We introduce the probabilities p_+,p_- of finding an up-spin and a down-spin, respectively,

$$N_+=\frac{N(1+m)}{2}\Rightarrow p_+=\frac{N_+}{N}=\frac{1+m}{2} \tag{11}$$

and

$$N_-=\frac{N(1-m)}{2}\Rightarrow p_-=\frac{N_-}{N}=\frac{1-m}{2}. \tag{12}$$

In one time step τ, three events are possible: the magnetization increases by $2/N$, decreases by $2/N$, or remains constant,

$$\gamma^+(m)=\mathrm{Prob}\left\{m\rightarrow m+\frac{2}{N}\right\},$$

$$\gamma^-(m)=\mathrm{Prob}\left\{m\rightarrow m-\frac{2}{N}\right\},$$

$$\gamma^0(m)=\mathrm{Prob}\{m\rightarrow m\}. \tag{13}$$

In this paper, we have introduced parameter T to the TC model in the following way: we flip spin $S_{i,j}$ with probability

$$p_{i,j}=\exp\left(\frac{-\Delta E_{i,j}}{T}\right), \tag{14}$$

where

$$\text{transition } \uparrow\uparrow\downarrow\rightarrow\uparrow\uparrow\uparrow \text{ gives } \Delta E=-2(1+\kappa)<0,$$

$$\text{transition } \uparrow\downarrow\downarrow\rightarrow\uparrow\downarrow\uparrow \text{ gives } \Delta E=2(1-\kappa)<0,$$

$$\text{transition} \downarrow\uparrow\downarrow\rightarrow\downarrow\uparrow\uparrow \text{ gives } \Delta E=2(\kappa-1)>0,$$

$$\text{transition } \downarrow\downarrow\downarrow\rightarrow\downarrow\downarrow\uparrow \text{ gives } \Delta E=2(1+\kappa)>0. \tag{15}$$

Thus, if we assume that $N\rightarrow\infty$, using the mean-field approach [16–18] we get the hopping probabilities

$$\gamma^+(m)=p_+p_+p_-+p_-p_+p_-+p_-^3\exp\left(-\frac{2(1+\kappa)}{T}\right)$$
$$+p_-^2p_+\exp\left(\frac{2(1-\kappa)}{T}\right),$$

$$\gamma^-(m)=p_+p_+p_-+p_-p_+p_-+p+-^3\exp\left(-\frac{2(1+\kappa)}{T}\right)$$
$$+p_+^2p_-\exp\left(\frac{2(1-\kappa)}{T}\right). \tag{16}$$

It is easy to notice that

$$\exp\left(\frac{2(1-\kappa)}{T}\right)=\exp\left(\frac{4}{T}\right)\exp\left(-\frac{2(1+\kappa)}{T}\right). \tag{17}$$

After simple algebraic transformations, we get

$$\gamma^+(m)=\frac{1-m^2}{4}+\frac{(1-m)^2}{8}\exp\left(-\frac{2(1+\kappa)}{T}\right)$$
$$\times\left[1-m+(1+m)\exp\left(\frac{4}{T}\right)\right],$$

$$\gamma^-(m)=\frac{1-m^2}{4}+\frac{(1+m)^2}{8}\exp\left(-\frac{2(1+\kappa)}{T}\right)$$
$$\times\left[1+m+(1-m)\exp\left(\frac{4}{T}\right)\right]. \tag{18}$$

It is easy to see that for $T\rightarrow 0$, both probabilities are equal,

$$\gamma^-(m)-\gamma^+(m)=0,$$

$$\gamma^-(m)+\gamma^+(m)=\frac{1-m^2}{2}. \tag{19}$$

The evolution of the magnetization can therefore be viewed as the motion of a random walker, as in the case of $T=0$ [16]. On the other hand, for $T\rightarrow\infty$ we get

$$\gamma^-(m)-\gamma^+(m)=m,$$

$$\gamma^-(m)+\gamma^+(m)=1. \tag{20}$$

This means that for a large value of T, the magnetization fluctuates around zero [due to $\gamma^-(m)-\gamma^+(m)=m$] and

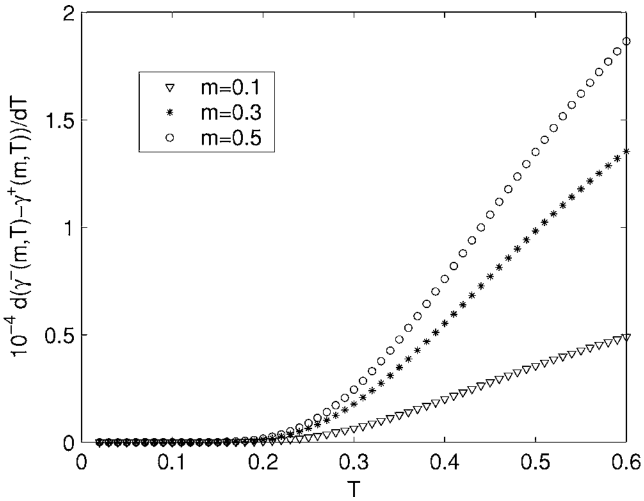

FIG. 9. Dependence between $\gamma^{-}(m)-\gamma^{+}(m)$ and T for several values of m.

changes in each time step [due to $\gamma^{-}(m)+\gamma^{+}(m)=1$].

The difference

$$\gamma^{-}(m)-\gamma^{+}(m)=\exp\left(-\frac{2(1+\kappa)}{T}\right)\frac{m}{2}\times\left[1+m+\frac{(1-m)^2}{2}\exp\left(\frac{4}{T}\right)\right] \quad (21)$$

for finite $T>0$ is plotted in Fig. 9. It is seen that this difference grows rapidly above $T=T^{*}\approx 0.2$. Below this value $\gamma^{-}(m)\approx\gamma^{+}(m)$, which is the case of $T\to 0$. We see that this mean-field result agrees quite well with simulations for which we have found $T^{*}\approx 0.19$.

V. SUMMARY

We have presented a generalized version of the two-component model. It is generalized in the sense that it admits parameter (T) in the following way: we flip spin $S_{i,j}$ with probability

$$p_{i,j}=\exp\left(\frac{-\Delta E_{i,j}}{T}\right), \quad (22)$$

where $\Delta E_{i,j}$ is the change of the disagreement function. In this paper, we have investigated the TC model in the degenerated phase, i.e., for $|\kappa|>1$. Four different structures which give the total number of 16 steady states have been observed for $T=0$ [13]. It turns out that for $T>0$, only two ferromagnetic states (all spins up or all spins down) are stable steady states. All other 14 states of type (b)–(d) are metastable.

The dynamics of the TC model for small $0<T<T^{*}$ is very interesting and consists of steady periods and rapid changes (transitions between two different types of trap states). For parameter $T>T^{*}$, the system is going directly to the ferromagnetic state.

In the constrained zero-temperature Glauber dynamics, the only possible moves are flips of isolated spins,

$$\uparrow\downarrow\uparrow\to\uparrow\uparrow\uparrow, \quad \downarrow\uparrow\downarrow\to\downarrow\downarrow\downarrow.$$

Each move suppresses two consecutive unsatisfied bonds. The system, therefore, eventually reaches a blocked configuration, where there are no isolated spins, i.e., up and down spins form clusters whose length is at least two. These blocked configurations are the zero-temperature analogs of metastable states. Thus the number of metastable states in the Ising system under Glauber dynamics grows with the system size and is infinite for the infinite system. In the TC model, the number of metastable states is equal to 14 independently of the system size. In this sense, the Ising system under Glauber dynamics is much richer than the TC model. On the other hand, all possible metastable states in the Glauber dynamics are of the same type—all contain ferromagnetic stripes, while in the TC model with $T=0$ there are four types of steady states (a)–(d) presented in Fig. 2.

[1] F. den Hollander, Stochastic Proc. Appl. **114**, 1 (2004).
[2] A. Bovier and F. Manzo, e-print cond-mat/0107376.
[3] C. Coulon, R. Clerac, L. Lecren, W. Wernsdorfer, and H. Miyasaka, e-print cond-mat/0404620.
[4] E. N. M. Cirillo and F. R. Nardi, J. Stat. Phys. **110**, 183 (2003).
[5] F. Schweitzer and J. A. Hołyst, Eur. Phys. J. B **15**, 723 (2000).
[6] G. De Smedt, C. Godreche, and J. M. Luck, Eur. Phys. J. B **27**, 363 (2002).
[7] G. De Smedt, C. Godreche, and J. M. Luck, Eur. Phys. J. B **32**, 215 (2003).
[8] C. Godreche and J. M. Luck, e-print cond-mat/0412077.
[9] V. Spirin, P. L. Krapivsky, and S. Redner, Phys. Rev. E **63**, 036118 (2001).
[10] V. Spirin, P. L. Krapivsky, and S. Redner, Phys. Rev. E **65**, 016119 (2002).
[11] J. L. Lebowitz and O. Penrose, J. Stat. Phys. **3**, 211 (1971).
[12] K. Sznajd-Weron, Phys. Rev. E **66**, 046131 (2002).
[13] K. Sznajd-Weron, Phys. Rev. E **70**, 037104 (2004).
[14] M. E. Fisher and W. Selke, Phys. Rev. Lett. **44**, 1502 (1980).
[15] D. Stauffer, Comput. Phys. Commun. **146**, 93 (2002).
[16] K. Sznajd-Weron, Phys. Rev. E **71**, 046110 (2005).
[17] F. Slanina and H. Lavicka, Eur. Phys. J. B **35**, 279 (2003).
[18] D. ben-Avraham, D. Considine, P. Meakin, S. Redner, and H. Takayasu, J. Phys. A **23**, 4297 (1990).